EPAPS supplementary material for "Majorana Fermions in Equilibrium and Driven Cold Atom Quantum Wires"

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In this supplement we perform a self-consistent calculation of the pairing energy of the 1D fermionic atoms surrounded by a 3D BEC of Feshbach molecules.

A. Interaction between Fermions and BEC

We consider the interaction

$$H_{int} = \sum_k g \int d^3 r \Psi^* (r)\, \eta_k (r)\, \eta_{-k} (r)\, a_{k,\uparrow} a_{-k,\downarrow} + h.c. \tag{1}$$

where g is the interaction strength, $\Psi(r)$ is the BEC wavefunction, $\eta_k(r)$ is the normalized wave function associated with the fermionic mode a_k. The dimensions are $[\Psi(r)] = [\eta_k(r)] = [L]^{3/2}$, and $[g] = [E]\,[L]^{3/2}$.

For cylindrical symmetric system, $\Psi(r) = \Psi(r_\perp)$ and $\eta_k(r) = \eta_{\|,k}(z)\,\eta_\perp(r_\perp)$, where $\eta_{\|,k}(z) = \frac{1}{\sqrt{L}} e^{ikz}$, $\eta_\perp(r_\perp) = \frac{1}{\sqrt{\pi} a_\perp} e^{-\frac{r_\perp^2}{2a_\perp^2}}$, L is the length and $a_\perp$ is the transverse confinement of the 1D tube of fermionic atoms. One can easily verify that $\int d^3 r\, |\eta_k(r)|^2 = 1$ and $\eta_k(r)\,\eta_{-k}(r) = \frac{1}{L}|\eta_\perp(r_\perp)|^2$ does not depend on k. The transverse confinement can be estimated as

$$a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}, \tag{2}$$

where m is the mass of the fermionic atom, $\omega_\perp$ is the transverse trap frequency. For ^{6}Li atoms with $\omega_\perp = 150\,(2\pi)\,kHz$, the transverse confinement is $a_\perp \approx 0.1\mu$m.

B. Pairing Energy

From the perspective of the 1D fermionic atoms, the interaction Hamiltonian is

$$H^F_{int} = \sum_k \Delta^* a_{k,\uparrow} a_{-k,\downarrow} + h.c. \tag{3}$$

with paring energy

$$\Delta = g \int d^3 r \Psi(r) \frac{1}{L} |\eta_\perp(r_\perp)|^2 . \tag{4}$$

When the interaction strength g is very small, H_{int} is just a perturbation and we can simply use the bulk BEC wavefunction $\Psi(r) \simeq \Psi(\infty) = \Xi$, which gives $\Delta \simeq g\Xi$. When the interaction strength g becomes large, H_{int} will modify the wavefunction of the BEC and $\Psi(r)$ will start to deviate from $\Psi(\infty)$, which will consequently change Δ as well. Hence we need to compute the wavefunction $\Psi(r)$ and the pairing energy Δ self-consistently.

We consider the fermionic system

$$H_F = H^F_{kin} + H^F_{int} \tag{5}$$

with kinetic energy $H_{kin}^F = \sum_{k,\sigma} \frac{k^2}{2m} a_{k,\sigma}^\dagger a_{k,\sigma}$. Using Bogoliubov transformation, we can obtain $\langle a_{k,\uparrow} a_{-k,\downarrow} \rangle = \frac{1}{2} \frac{\Delta}{\sqrt{\left(\frac{k^2}{2m}\right)^2 + \Delta^2}}$. Then we can compute the quantity

$$\sum_k \langle a_{k,\uparrow} a_{-k,\downarrow} \rangle = \frac{L}{h} \int dk \frac{1}{2} \frac{\Delta}{\sqrt{\left(\frac{k^2}{2m}\right)^2 + \Delta^2}} \approx 0.92 \frac{L}{h} \sqrt{2m\Delta}. \tag{6}$$

C. BEC Wavefunction

From the perspective of the 3D BEC, the interaction Hamiltonian is

$$H_{int}^B = \sum_k g \int d^3 r \Psi^* (r) \eta_k (r) \eta_{-k} (r) \langle a_{k,\uparrow} a_{-k,\downarrow} \rangle + h.c. \tag{7}$$

$$= \int d^3 r \left[\Psi^* (r) Q (r) + h.c. \right] \tag{8}$$

with

$$Q (r) = \sum_k g \eta_k (r) \eta_{-k} (r) \langle a_{k,\uparrow} a_{-k,\downarrow} \rangle$$

$$\approx 0.92 \frac{g \ e^{-r_\perp^2 / a_\perp^2}}{\pi h a_\perp^2} \sqrt{2m\Delta}. \tag{9}$$

Hence the BEC has energy

$$E = \int d^3 r \left[\frac{\hbar^2}{2 (2m)} \left| \triangledown \Psi (r) \right|^2 + \frac{1}{2} U_0 \left| \Psi (r) \right|^4 \right] + \int d^3 r \left[\Psi^* (r) Q (r) + h.c. \right], \tag{10}$$

where $2m$ is the mass of the Feshbach molecules and U_0 is the on-site interaction between molecules. Suppose the BEC has chemical potential μ and particle number $N = \int d^3 r \left| \Psi (r) \right|^2$, the requirement $\frac{\delta (E - \mu N)}{\delta \Psi^* (r)} = 0$ gives the modified Gross-Pitaevskii equation:

$$-\frac{\hbar^2}{2 (2m)} \triangledown^2 \Psi (r) + U_0 \left| \Psi (r) \right|^2 \Psi (r) + Q (r) = \mu \Psi (r). \tag{11}$$

The remaining task is to solve the non-linear differential equations [Eqs.(4,9,11)] self-consistently.

D. Self-consistent Solution

It will be convenient to convert into dimensionless quantities. The healing length of BEC is $\xi = \frac{\hbar}{\sqrt{2(2m)\mu}}$, the bulk density of the BEC is $n = \frac{\mu}{U_0}$, and the bulk amplitude is $\Psi (r_\perp \to \infty) = \sqrt{n}$. We may introduce the dimensionless transverse radius $\tilde{r} \equiv \frac{r_\perp}{\xi}$, the rescaled BEC wavefunction $f (\xi \tilde{r}) \equiv \frac{\Psi(\xi \tilde{r})}{\Psi(\infty)}$, the transverse wavefunction of the fermionic mode $\tilde{\eta} (\tilde{r}) = \frac{(\xi / a_\perp)^2}{\pi} e^{-(\tilde{r} \xi / a_\perp)^2}$, and the dimensionless back-action $q (\xi \tilde{r}) \equiv \frac{Q(\xi \tilde{r})}{\Psi(\infty) \mu} = q_0 \tilde{\eta} (\tilde{r})$ with $q_0 = \frac{g \sqrt{2m\Delta / n}}{\mu h \xi^2}$. We can rewrite Eqs.(4,9,11) in terms of the dimensionless functions:

$$\Delta = g n^{1/2} \int d^2 \tilde{r} f (\tilde{r}) \frac{1}{\pi (a_\perp / \xi)^2} e^{-(\tilde{r} \xi / a_\perp)^2}. \tag{12}$$

$$q_0 = \frac{g \sqrt{2m\Delta / n}}{\mu h \xi^2} \tag{13}$$

$$-\frac{1}{\tilde{r}} \frac{d}{d\tilde{r}} \tilde{r} \frac{d}{d\tilde{r}} f (\tilde{r}) + f^3 (\tilde{r}) + q_0 \tilde{\eta} (\tilde{r}) = f (\tilde{r}). \tag{14}$$

Parameter	U_0	μ	n	m	$a_\perp$	ξ	g_0
Value	$5\times10^{-11}\mathrm{K}\cdot\mu\mathrm{m}^3$	$5\times10^{-9}\mathrm{K}$	$10^{14}\mathrm{cm}^{-3}$	$6\times1.67\times10^{-27}\mathrm{kg}$	$0.1\mu\mathrm{m}$	$2\mu\mathrm{m}$	$1\times10^{-7}\mathrm{K}\cdot\mu\mathrm{m}^{3/2}$

TABLE I: Typical choice of parameters.

The boundary conditions are $\left.\frac{df(\tilde{r})}{d\tilde{r}}\right|_{\tilde{r}=0}=0$ and $f(\tilde{r})|_{\tilde{r}\to\infty}=1$. For given parameters of $\{U_0,\mu,m,a_\perp,\xi\}$, we can numerically compute $f(\tilde{r})$ and Δ for different values of g.

For $\xi>a_\perp$, the characteristic value of g can be determined by assuming $\Delta=g_0n^{1/2}$ and $q_0=1$. That is,

$$q_0|_{g=g_0,\Delta=g_0n^{1/2}}=1 \tag{15}$$

which gives

$$g_0=1.3\frac{\hbar^2n^{1/6}}{m}. \tag{16}$$

The typical choice of parameters are listed in Table I. Using this set of typical choice of parameters, we numerically compute $\Delta=\Delta(g)$ which has a maximum value of $\Delta\approx25\,(2\pi)\,kHz$ at $g\approx2.5g_0$ as shown in Fig. 1a. We can also obtain $f(\tilde{r})$ for different choices of g as shown in Fig. 1b. For large g, the BEC wavefunction amplitude is significantly reduced in the neighborhood of trapped fermions.

E. Rough Estimate

We can also perform a rough estimate by computing

$$\Delta_{est}=g_0n^{1/2}\approx\frac{\hbar^2n^{2/3}}{m} \tag{17}$$

For a typical choice of parameters, we have the estimated value of $\Delta_{est}\approx40\,(2\pi)\mathrm{kHz}$, which agrees fairly well with the self-consistent calculation.

F. Summary

In conclusion, we have performed a self-consistent calculation of the pair energy for the 1D fermionic atoms surrounded by a 3D BEC of Feshbach molecules. We find that for practical choice of parameters listed in Table I, the pairing energy can can be as large as $25\,(2\pi)\mathrm{kHz}$ as shown in Fig. 1a.

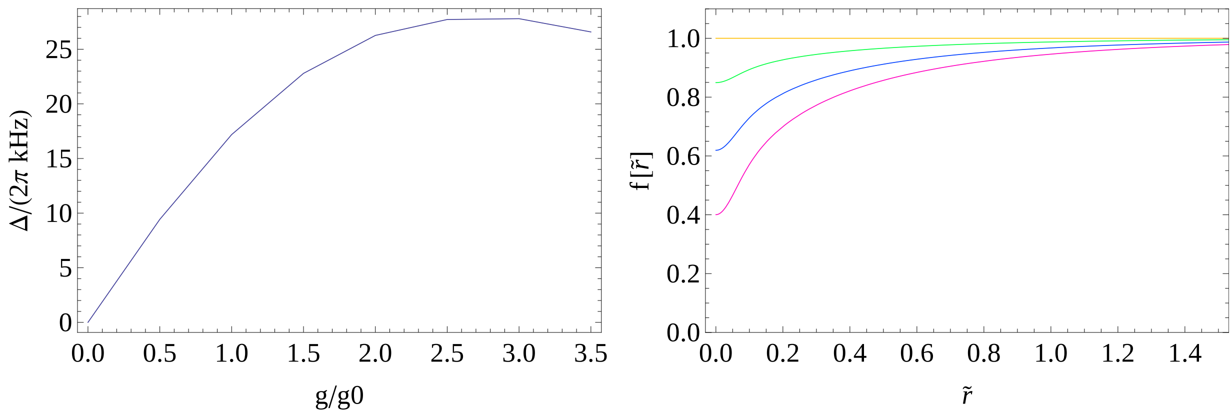

FIG. 1: (a) The pairing gap $\Delta=\Delta(g)$ as a function of the interaction strength g. (b) The rescaled BEC wavefunction $f(\tilde{r})$ for $g/g_0=0,1,2,3$ (from top to bottom).